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Issues of duality on noncommutative manifolds: The nonequivalence between self-dual and topologically massive models

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We study issues of duality and dual equivalence in noncommutative manifolds. In particular, the question of dual equivalence for the actions of the noncommutative extensions of the self-dual model in 3D space-time and the Maxwell-Chern-Simons model is investigated. We show that former model *is not* dual equivalent to the noncommutative extension of the Maxwell-Chern-Simons model, as widely believed, but a deformed version of it that is disclosed here. Our results are not restricted to any finite order in the Seiberg-Witten expansion involving the noncommutative parameter θ.

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This paper is devoted to study the notions of duality transformations, self-duality, and duality equivalence on noncommutative manifolds. This seems necessary because noncommutativity introduces new elements in the analysis that demand further investigation regarding already known phenomena and the eventual appearance of new physics. Recent developments in string theory and mathematics have motivated the study of field theory models constructed on noncommutative space-time. Field theory models constructed on such spaces have many interesting features which their commutative counterparts do not share, like the possibility of novel soliton solutions, UV/IR mixing, and loss of the duality correspondence basically because of the presence of nonlinear and nonlocal terms introduced by noncommutativity of the space-time.

The study of duality is done here using elements of the dual projection approach—a canonical transformation that separates the dynamical sector of a given theory from the sector that displays the symmetry of the action. This approach has proven useful in disclosing duality mapping and model equivalence in the past [1,2], in a different context. In Ref. [1] in particular, it has been shown that, in the ordinary commutative space, the Maxwell-Chern-Simons action (MCS) can be diagonalized in two sectors: One sector is described by the self-dual (SD) model that carries the self-dual dynamics known to exist in the MCS model but is not gauge invariant, while the other sector is described by a pure Chern-Simons (CS) theory that, due to its topological character, has no propagating degree of freedom but carries the gauge symmetry of the original model. This procedure is here adapted to study duality in noncommutative manifolds.

Our approach is not based on the perturbative structure of the Seiberg-Witten map. Consequently, our final answer results to be nonperturbative in the noncommutative constant θ. This is distinct to other approaches until now. We clearly show that the expected and largely believed dual equivalence between a noncommutative extension of the self-dual model (NC-SD) and the noncommutative version of the topologically massive theory (NC-MCS) is not realized in the conditions considered here. In order to support our findings, we compute explicitly the gauge invariant, second-order model dual to the NC-SD model.

The investigation in noncommutative field theory has experienced an outburst of interest after the work of Seiberg and Witten [3] in open string and D-brane physics. A great deal of effort is still under way in order to compute the relevant modification and interpret the physical consequences that space noncommutativity effects have over known phenomena. Among the topics under investigation, it becomes quite important to understand to which degree duality is preserved or modified under the restriction imposed by noncommutativity. Duality, as is well known, is a symmetry concept that is very useful in different areas of physics with dramatic consequences, quite particularly in gauge field theories where it has been used to establish (i) dual equivalence between the solitonic solution of a given model with the particle excitations of its dual and (ii) to disclose the presence of the hidden gauge symmetry in self-dual solutions of some planar models, as shown by Deser and Jackiw [4], with consequences in the long distance physics of massive gauge theories. It is therefore of fundamental importance, in order to extend the use of these concepts to this new arena, to study the presence of duality in noncommutative manifolds since it acts upon the theory in a quite distinct manner.

The crucial point seems to be the Seiberg-Witten map [3], since duality has been approached exclusively upon this concept [5]. The rationale behind this technique is the mapping of the noncommutative gauge fields into ordinary commutative ones in powers of the noncommutative parameter θ in this way establishing a connection between the gauge orbits in both regimes and has been extended to tensors of distinct nature [6]. This idea, originally applied in the study of duality in NC electromagnetic phenomena [5], was extended to establish the duality equivalence between different mathematical descriptions of some known planar phenomena in ordinary 3D space-time [7].

It is shown in the present report, via the dual projection technique, that a simple and beautiful solution of the duality problem that avoids the use of the Seiberg-Witten expansion is possible. Besides the nonperturbative character of the solution, this approach which is derived from a Noether embedding algorithm, makes no use of a gauge invariant master action or the Lagrange multiplier imposing zero-curvature condition approach to establish duality.

Let us consider the Maxwell-Chern-Simons action, defined over a noncommutative manifold (for a review of the noncommutative field theory, see [8,9])

$$S[A] = \int d^3x \Big[-\frac{1}{2} F_\mu \star F^\mu + \frac{m}{2} \epsilon^{\mu\nu\lambda} \Big(A_\mu \star \partial_\nu A_\lambda + \frac{2i}{3} A_\mu \star A_\nu \star A_\lambda \Big) \Big], \quad (1)$$

where m is a free parameter with mass dimensions. We used $F_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda} F^{\nu\lambda}$ and $F^{\nu\lambda} = \partial^{[\nu} A^{\lambda]} + i[A_\nu, A_\lambda]_\star$. The $\star$ product is given by the Moyal formula,

$$(f \star g)(x) = e^{(i/2)\theta_{\mu\nu}\partial_{\xi_\mu}\partial_{\zeta_\nu}} f(x+\xi) g(x+\zeta)|_{\xi=\zeta=0}, \quad (2)$$

and the noncommutativity of the space-time is described by

$$[x^\rho, x^\sigma]_\star = i\theta^{\rho\sigma}. \quad (3)$$

The physics behind the presence of the noncommutative Chern-Simons term has been extensively studied in recent years [10–19].

Our goal is, as discussed in the introduction above, to look for the NC dual of the theory (1). To apply the dual projection algorithm into the above model, one needs first to include an ancillary field, say π_μ, to duplicate the dimension of the phase space or, equivalently, lower the order of the differential equations. The Lagrangian density for the NC-MCS model becomes

$$\mathcal{L}[A] = \Big[\frac{m^2}{8} (\pi_\mu - A_\mu)^2_\star + \frac{m}{2} \epsilon^{\mu\nu\lambda} \pi_\mu \star \partial_\nu A_\lambda + \frac{mi}{6} \epsilon^{\mu\nu\lambda} (3\pi_\mu - A_\mu) \star A_\nu \star A_\lambda \Big]. \quad (4)$$

As shown in [1], the final effect of the dual projection algorithm is materialized by the following canonical transformation that transforms the phase space of this model back into the configuration space of two-field theory, albeit of first order,

$$\pi_\mu = f^+_\mu + f^-_\mu; \qquad A_\mu = f^+_\mu - f^-_\mu. \quad (5)$$

Plugging this field redefinition back into NC-MCS action, we obtain

$$\mathcal{L}[A] = \frac{m}{2} \Big[\epsilon^{\mu\nu\lambda} \Big(f^+_\mu \star \partial_\nu f^+_\lambda + \frac{2i}{3} f^+_\mu \star f^+_\nu \star f^+_\lambda \Big) + m (f^-_\mu)^2_\star - \epsilon^{\mu\nu\lambda} \Big(f^-_\mu \star \partial_\nu f^-_\lambda - \frac{4i}{3} f^-_\mu \star f^-_\nu \star f^-_\lambda \Big) - i2\epsilon^{\mu\nu\lambda} (f^-_\mu \star f^-_\nu \star f^+_\lambda) \Big]. \quad (6)$$

Therefore, following the steps of Ref. [1], we have rewritten the NC-MCS in terms of two components fields, $f^\pm_\mu$, presenting the desired features: a pure Chern-Simons term for the f^+_μ, and a self-dual-like model composed of a mass term and a Chern-Simons termlike term. There are, however, important issues to overcome. One is that the cubic factor in the CS-like term has the wrong coefficient. The second and most important difficulty is the presence of a mixed cubic term [the last term in (6)]. It shows that the original NC-MCS cannot be diagonalized in a dynamical sector plus a topological sector displaying the original symmetry which seems to be a hard nut to crack. Therefore, differently from the commutative case, here the NC-MCS theory *is not* dual to the NC-SD theory. This is the first part of our result. Next we try to fix the above-mentioned problems.

It is important to observe at this juncture that a deformation of the Chern-Simons term is of no help to solve any of the above difficulties. A close inspection in expression (4) shows that the cubic terms originate from two contributions: an $A^3_\star$ term and a $\pi \star A^2_\star$ term. A more symmetric combination seems to be more adequate and, indeed, a little theoretical experimentation shows that the inclusion of another cubic term in the form of $i\alpha A \star \pi^2_\star$, with α being an adjustable constant tensor, is able to cure both sicknesses at once. It should be observed that the presence of such a term demands a profound modification on the structure of the theory since the substitution $\pi_\mu \star \pi^\mu \to \pi_\mu \star (1 + i\alpha A)^{\mu\nu} \star \pi_\nu$ requires an initial structure in the second-order Lagrangian density as $F_\mu \star F^\mu \to F^\mu \star (1 + i\alpha A)^{-1}_{\mu\nu} \star F^\nu$. The deformation therefore should be in the Maxwell part.

Let us then consider the following Lagrangian density for a *deformed* NC-MCS theory:

$$\mathcal{L}[A] = \Big[-\frac{1}{2} F^\mu \star ({}^*M_{\mu\nu}) \star F^\nu + \frac{m}{2} \epsilon^{\mu\nu\lambda} \Big(A_\mu \star \partial_\nu A_\lambda + \frac{2i}{3} A_\mu \star A_\nu \star A_\lambda \Big) \Big]. \quad (7)$$

The first-order Lagrangian density for the deformed NC-MCS model becomes then

$$\mathcal{L}[A] = \Big\{\frac{m^2}{8}(\pi - A)^\mu \star (^*M_{\mu\nu})^{-1} \star (\pi - A)^\nu + \frac{m}{2}\epsilon^{\mu\nu\lambda}\Big[\pi_\mu \star \partial_\nu A_\lambda + \frac{i}{3}(3\pi - A)_\mu \star A_\nu \star A_\lambda\Big]\Big\},$$

where

$$(^*M_{\mu\nu})^{-1} = \delta_{\mu\nu} - \frac{2i}{m}\epsilon_{\mu\nu\lambda}A^\lambda. \tag{8}$$

Following the previous steps and plugging the field redefinition (5) back into the deformed NC-MCS action, we obtain

$$\mathcal{L}[A] = \frac{m}{2}\Big[\epsilon^{\mu\nu\lambda}\Big(f^+_\mu \star \partial_\nu f^+_\lambda + \frac{2i}{3}f^+_\mu \star f^+_\nu \star f^+_\lambda\Big) + m(f^-_\mu)^2_\star - \epsilon^{\mu\nu\lambda}\Big(f^-_\mu \star \partial_\nu f^-_\lambda + \frac{2i}{3}f^-_\mu \star f^-_\nu \star f^-_\lambda\Big)\Big],$$

which is our main result. The first line displays a pure NC-CS term while the second line shows a NC-SD model displaying the same helicity of the original NC-MCS theory. Clearly, the original gauge symmetry is supported by the first sector while dynamics is manifest by the second as discussed above. Therefore we were able to obtain the thoroughly searched dual of the NC-SD model as manifested by the *deformed* NC-MCS, not the original theory. It is important to mention that this is an exact result.

This result is a very important contribution to the present stage of investigation on this subject since it shines light on this debated matter by bringing in an exact result for an interesting model and a new procedure. It is also of importance to the bosonization program in the NC space. As shown in [20], it is possible to establish a connection between the massive Thirring model and the NC-SD model, at least in the long wavelength limit. Our results clearly show that, contrary to what would be naively expected, there is no connection from this model to the NC-MCS model. We showed that NC-MTM is, in fact, dual to the model proposed in (7). Besides, since non-Abelian gauge theories have a similar gauge structure as the noncommutative gauge theories, the study of duality and bosonization in noncommutative space-times is also of interest as the studies of the latter may shed further light on similar problems in the former.

In summary, we have directly computed the dual equivalent of the NC-SD model. Our results show that the NC-SD model *is not* dual to the NC-MCS model as thoroughly searched but to a *deformed* version of it as presented in Eq. (7). The diagonalization of this action was possible and displays the presence of a pure NC-CS term carrying the gauge symmetry plus a dynamical sector in the form of a NC-SD model. Our result clearly explains the reasons for the failure in obtaining duality equivalence through the master action approach and other methods, being limited therefore to the first-order expansion in the Seiberg-Witten map. Although we have found it in a very particular model, it seems to be a quite general phenomenon in NC manifolds. This is a new result that illuminates the discussion on the subject and may have further consequences on the studies of duality on NC manifolds.

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